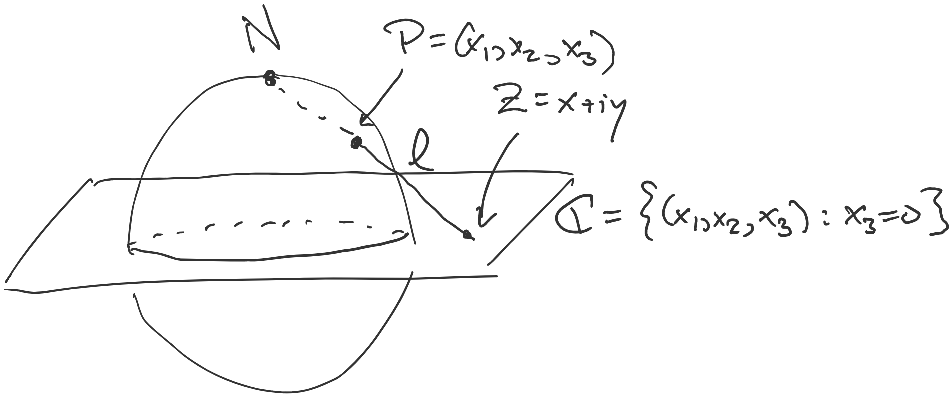
# Lecture 1

Thursday, September 26, 2019 4:06 PM

The Riemann sphere / extended plane / complex projective space.

The extended plane $\mathbb{C}_\infty$ as a set is $\mathbb{C}\cup\{\infty\}$. We identify it w/ $\underset{\{x_1^2+x_2^2+x_3^2=1\}}{S^2} \subseteq \mathbb{R}^3$:




$$l = \{\ t(0,0,1) + (1-t)(x,y,0)\}. \quad P = l \cap S^2 \Rightarrow$$

$$\|((1-t)x, (1-t)y, t)\|^2 = 1 \Rightarrow (1-t)^2\underbrace{(x^2+y^2)}_{|z|^2} + t^2 = 1 \Rightarrow$$

$$1-t^2 = (1-t)^2|z|^2 \Rightarrow t^2(|z|^2+1) - 2t|z|^2 + |z|^2 - 1 = 0$$

$$t^2 - \frac{2|z|^2}{|z|^2+1}t + \frac{|z|^2-1}{|z|^2+1} = 0 \qquad t = \frac{|z|^2}{|z|^2+1} \pm \sqrt{\underbrace{\frac{|z|^4}{(|z|^2+1)^2} - \frac{|z|^2-1}{|z|^2+1}}_{\frac{|z|^4 - (|z|^2-1)(|z|^2+1)}{(|z|^2+1)^2} = \frac{1}{(|z|^2+1)^2}}}$$

$|z|^4 - 1$

$$\Rightarrow \quad t = \frac{|z|^2(\pm)1}{|z|^2+1} = \frac{|z|^2-1}{|z|^2+1}$$

- So $P = (x_1, x_2, x_3) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$ - Stereographic projection.

$P = N$ if $z = \infty$.

- Conversely, given $P = (x_1, x_2, x_3) \in S^2$, we find $z = x+iy$ by noting that in the line $l$ above, $t = x_3$. We the solve for $x, y$ and obtain $z = \frac{x_1 + ix_2}{1-x_3}$.

<u>Fubini-Study metric/distance on $\mathbb{C}_\infty$.</u>

... identifying two points

Fubini-Study metric / distance on $\mathbb{C}_\infty$.

We define a distance function on $\mathbb{C}_\infty$ by identifying two points $z = x+iy$, $z' = x'+iy'$ w/ their corresponding points $P, P' \in S^2$ and taking the standard Euclidian distance between $P, P'$, i.e.

$$d_\infty(z,z') = \left[(x_1-x_1')^2 + (x_2-x_2')^2 + (x_3-x_3')^2\right]^{1/2}; \quad P = (x_1, x_2, x_3),\ P' = (x_1', x_2', x_3').$$

Tracing back through the formulas above, we get

$$d_\infty(z,z') = \frac{2|z-z'|}{\left[(1+|z|^2)(1+|z'|^2)\right]^{1/2}}.$$

For $z' = \infty$, we get $d_\infty(z,\infty) = \dfrac{2}{(1+|z|^2)^{1/2}}$.